An Exchange Identity for Non-linear Fields

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November 28, 2005

Abstract

We establish a useful identity for intertwining a creation or annihilation operator with the heat kernel of a self-interacting bosonic field theory.

I Background

Consider creation operators $a^*(f)$ and annihilation operators $a(h)$, both linear in their respective test functions $f, h \in L^2(\mathbb{R}, dx)$, acting on the Fock Hilbert space $\mathcal{H}$, and satisfying the canonical commutation relations

$$[a(h), a^*(f)] = \left\langle \bar{h}, f \right\rangle_{L^2} \ . \tag{I.1}$$

The free field Hamiltonian H_0 acts on $\mathcal{H}$ and also on the one particle subspace $L^2(\mathbb{R}, dx)$, where one denotes its action by the operator $\omega = (-d^2/dx^2 + m^2)^{1/2}$. The time-zero field $\varphi(g)$ has the definition $\varphi(g) = a^*\left((2\omega)^{-1/2}g\right) + a\left((2\omega)^{-1/2}g\right)$. The operators a^*, a, and H_0 satisfy the relation

$$e^{a(h)} e^{-\beta H_0} e^{a^*(f)} = e^{\left\langle \bar{h}, e^{-\beta\omega} f\right\rangle} e^{a^*(e^{-\beta\omega} f)} e^{-\beta H_0} e^{a(e^{-\beta\omega} h)} \ . \tag{I.2}$$

This identity (in case either $f = 0$ or $h = 0$) is known in the constructive field theory literature as a "pull-through" identity.

The pull-through identity played a central role in the analysis of properties of heat kernels for field theories with interaction. It provided a fundamental ingredient in the analysis of the domain of the fields, in the proof of the cluster expansion, in the proof of the existence of a mass gap, and especially in the proof of the existence of an upper mass gap in weakly-coupled $\lambda\mathcal{P}(\varphi)$ quantum field models, see [3, 4]. An introduction to this work can be found in [2, 5], but one must visit the original literature for details. The free-field pull-trough identity provides a key step in the proof of the nuclearity property for the *free* field by Buchholz and Wichmann [1], and motivates finding the related identity (II.16) for a field theory with a $\mathcal{P}(\varphi)$ polynomial interaction.

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II The Main Result

In this paper we give a new identity similar to (I.2), but with H_0 replaced by the Hamiltonian H for a non-linear field theory (with a spatial cutoff). Because of the non-linearity, the Hamiltonian on the left of the identity differs from the Hamiltonian on the right. Remarkably, we present a closed form for the relationship between the time-dependent Hamiltonians.

At least one of the Hamiltonians must depend on time, so we allow both to do so (in a particular way) and denote the time-dependent Hamiltonians that arise by $\mathbf{H}(s)$. One must replace the semigroup $e^{-\beta H}$ by the time-ordered exponential $T\exp\left(-\int_0^\beta \mathbf{H}(s)ds\right)$, where we use the convention that time increases from left to right. Call the resulting identity that generalizes (I.2) an *exchange identity.* It has the structure

$$e^{a(h)}\left(Te^{-\int_0^\beta \mathbf{H}_1(s)ds}\right)e^{a^*(f)} = e^{\left\langle \bar{h}, e^{-\beta\omega}f\right\rangle}e^{a^*(e^{-\beta\omega}f)}\left(Te^{-\int_0^\beta \mathbf{H}_2(s)ds}\right)e^{a(e^{-\beta\omega}h)}\ . \tag{II.1}$$

We give the explicit form of $\mathbf{H}_1(s)$ and $\mathbf{H}_2(s)$ in Theorem II.1.

In this paper we emphasize the algebraic structure of the exchange identity. We do not analyze the convergence of exponential series or the convergence of families of such series. We expect that most such questions in specific applications of interest can be addressed by the reader—hopefully without undue difficulty. In order to ensure stability we do assume that the basic interaction polynomial is bounded from below. In order to avoid infra-red problems we also assume that the mass of H_0 is strictly positive, or else we work with a twist field defined on a spatial circle. All in all, the complete justification of Theorem II.1, even for an elementary non-linearity, requires the introduction and removal of an ultra-violet cutoff, using for instance, a Feynman-Kac representation and estimates on path space to establish stability bounds and convergence of associated vectors and operators. See the methods in [2]. Once one establishes the basic stability bound in a particular example—uniform in the ultra-violet cutoff—details concerning convergence of vectors and operators, domains on which Theorem II.1 applies, etc., will all fall into place. The case of complex functions f or h leads to non-hermitian Hamiltonians $\mathbf{H}_1$ or $\mathbf{H}_2$. But these always arise as small non-hermitian perturbations of a self-adjoint Hamiltonian, so standard methods should apply.

While these steps need to be carried out in particular examples, including such details here would obscure the simplicity of the presentation of our new identity. This elegant form of the exchange identity raises the question whether one might make progress toward finding other useful closed-form expressions in the solution of $\mathcal{P}(\varphi)_2$ quantum field theories.

II.1 Interactions

The usual interaction polynomial arises from a polynomial $\mathcal{P}(\xi)$ and is defined as

$$H_I(\mathcal{P},\lambda) = \int :\mathcal{P}(\varphi(x)):\lambda(x)dx\ , \qquad \text{where } \mathcal{P}(\xi) = \xi^{2k} + \sum_{j=0}^{2k-1} c_j\xi^j\ , \tag{II.2}$$

where $:\mathcal{P}(\varphi(x)):$ is the normal-ordered energy density, see for example [2]. We take the spatially dependent cutoff $0 \leq \lambda(x)$ to be smooth and compactly supported. This cutoff defines an interesting class of polynomial interactions.

Let us now generalize this form of interaction, by assigning a spatially-dependent coupling constant $\lambda\mathbf{g}_j(x)$ to the j^{th}-derivative $\mathcal{P}^{(j)}$ of the polynomial $\mathcal{P}$. Write

$$\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{g}) := \sum_{j=0}^{\infty} H_I(\mathcal{P}^{(j)}, \lambda\mathbf{g}_j) \ . \tag{II.3}$$

The sum in (II.3) terminates with $j = 2k$, the degree of $\mathcal{P}$. Consider now $\mathbf{g}$ as a vector of coupling constants, with components $\mathbf{g}_j$.

Motivated by this form of interaction, define a vector space of sequences of complex-valued, bounded functions on $\mathbb{R}$. These vectors $\mathbf{f} \in \mathcal{C}$ have components $\mathbf{f}_j(x)$, $j \in \mathbb{Z}_+$. There is a natural scalar multiplication by smooth functions $\lambda(x)$,

$$(\lambda\mathbf{f})_j(x) = \lambda(x)\mathbf{f}_j(x) \ . \tag{II.4}$$

There is also a natural imbedding $\iota : L^2(\mathbb{R}) \mapsto \mathcal{C}$ given by

$$\iota(f) = \{0, f, 0, \ldots\} \ . \tag{II.5}$$

In addition to multiplication (II.4) by scalars, the vector space $\mathcal{C}$ is a commutative ring with the product $* : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by

$$(\mathbf{f} * \mathbf{g})_j(x) = \sum_{k=0}^{j} \mathbf{f}_k(x)\mathbf{g}_{j-k}(x) \ . \tag{II.6}$$

The identity in $\mathcal{C}$ is the function

$$\mathrm{Id} = \{1, 0 \ldots\} \ , \tag{II.7}$$

and the n^{th} $*$-power of $\iota(f)$ is

$$\iota(f) * \iota(f) * \cdots * \iota(f) = \{0, 0, \ldots, f(x)^n, \ldots\} \ . \tag{II.8}$$

Also define $\iota(f)^0 = \mathrm{Id}$. In terms of these powers, there is a natural exponential imbedding $\Gamma : f \mapsto \mathcal{C}$ given by

$$\mathbf{\Gamma}(f) = e^{\iota(f)} = \mathrm{Id} + \sum_{j=1}^{\infty} \frac{1}{j!}\, \iota(f)^j = \{1, f(x), \frac{1}{2!}f(x)^2, \ldots, \frac{1}{j!}f(x)^j, \ldots\} \ . \tag{II.9}$$

With this notation,

$$\mathbf{\Gamma}(f) * \mathbf{\Gamma}(g) = \mathbf{\Gamma}(f+g) \ , \qquad \mathbf{\Gamma}(f)^{-1} = \mathbf{\Gamma}(-f) \ , \qquad \text{and } \mathbf{\Gamma}(0) = \mathrm{Id} \ . \tag{II.10}$$

The special case $H_I(\mathcal{P}, \lambda)$ of (II.2) corresponds to $\mathbf{g} = \mathrm{Id} = \mathbf{\Gamma}(0)$. We use a bold-face Hamiltonian to denote one determined by a polynomial $\mathcal{P}$ (bounded from below) as well as its derivatives $\mathcal{P}^{(j)}$ in the fashion (II.3) with

$$\mathbf{g} = \mathbf{\Gamma}(g) \ . \tag{II.11}$$

In the following we find that perturbations of this type play a special role, especially when g has the form $f_s + h_{\beta-s}$, where

$$f_s(x) = \left((2\omega)^{-1/2}\, e^{-s\omega} f\right)(x)\ . \tag{II.12}$$

(Note $f_0 \neq f$.) Therefore consider the time-dependent, total Hamiltonians at time s of the form

$$\mathbf{H} = \mathbf{H}(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s + h_{\beta-s})) = H_0 + \mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s + h_{\beta-s}))\ , \tag{II.13}$$

(where the vacuum energy has not been renormalized to zero).

An elementary pull-through identity has the form

$$e^{a(h)}\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s))ds}\right) = \left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s+h_{\beta-s}))\,ds}\right) e^{a(e^{-\beta\omega}h)}\ . \tag{II.14}$$

We establish this and related identities in the next section.

II.2 Exchange Identities

The following generalization states how to exchange the position of the product of an exponential of a creation and an exponential of an annihilation operator.

Theorem II.1. (Exchange Identity) *As a formal identity,*

$$e^{a(h)}\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s))ds}\right) e^{a^*(f)} = e^{\left\langle \bar{h}, e^{-\beta\omega} f\right\rangle} e^{a^*(e^{-\beta\omega}f)}\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(f_s+g_s+h_{\beta-s})ds}\right) e^{a(e^{-\beta\omega}h)}\ . \tag{II.15}$$

Remark. The exchange identity (II.15) reduces to the pull-through identity (II.14) for $f = 0$. Furthermore, the special choice $\mathbf{g} = \mathrm{Id}$ gives

$$e^{a(h)} e^{-\beta(H_0 + H_I(\mathcal{P},\lambda))} e^{a^*(f)} = e^{\left\langle \bar{h}, e^{-\beta\omega} f\right\rangle} e^{a^*(e^{-\beta\omega}f)}\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(f_s+h_{\beta-s})ds}\right) e^{a(e^{-\beta\omega}h)}\ . \tag{II.16}$$

This special case shows that if one begins with a time-independent interaction, the exchange identity gives rise to a time-dependent Hamiltonian. After the exchange, the perturbation of the original Hamiltonian involves perturbations of lower degree than $\mathcal{P}$, and the coupling constant of the highest degree term is unchanged. Therefore the standard stability bounds of constructive quantum field theory (based on the Feynman-Kac formula) should yield the existence of the time ordered exponential $\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(f_s+h_{\beta-s})ds}\right)$ of the time-dependent Hamiltonian.

Lemma II.2. *Let $t_1 \leq t_2$. Consider the Hamiltonian $\mathbf{H}(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s))$ and the time-ordered exponential*

$$R(t_2, t_1) = Te^{-\int_{t_1}^{t_2} \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s))ds}\ , \tag{II.17}$$

with time increasing from left to right. Then $R(t_2, t_1)$ is the solution to the differential equation

$$\frac{\partial}{\partial t_2} R(t_2, t_1) = -\mathbf{H}(\mathcal{P}, \lambda\mathbf{\Gamma}(g_{t_2}))R(t_2, t_1)\ , \qquad \text{with } R(t,t) = I\ , \tag{II.18}$$

as well as the equation

$$\frac{\partial}{\partial t_1} R(t_2, t_1) = R(t_2, t_1) \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(g_{t_1})) \,, \qquad \textit{with } R(t,t) = I \,. \tag{II.19}$$

Proof. Assume that the time-ordered exponential (II.17) can be expanded according to usual perturbation series. Integrating the relation (II.18) gives

$$\begin{aligned} R(t_2,t_1) &= I - \int_{t_1}^{t_2} ds_1 \mathbf{H}(s_1) R(s_1,t_1) \\ &= I - \int_{t_1}^{t_2} ds_1 \mathbf{H}(s_1) + \int_{t_1}^{t_2} ds_1 \int_{t_1}^{s_1} ds_2\, \mathbf{H}(s_1)\mathbf{H}(s_2) R(s_2,t_1) \\ &= \cdots = \sum_{j=0}^{\infty} (-1)^j \int_{t_1 \le s_j \cdots \le s_2 \le s_1 \le t_2} ds_1 \cdots ds_j\, \mathbf{H}(s_1) \cdots \mathbf{H}(s_j) \\ &= T e^{-\int_{t_1}^{t_2} \mathbf{H}(s) ds} \,. \end{aligned} \tag{II.20}$$

This also shows that $R(t_2 + \epsilon, t_1) - R(t_2, t_1) \sim -\epsilon H(t_2) R(t_2, t_1)$. One completes the proof that the time-ordered exponential satisfies the equation (II.18) by removing the regularization and establishing convergence of the approximation.

A similar iteration gives

$$\begin{aligned} R(t_2,t_1) &= I - \int_{t_1}^{t_2} ds_1 R(t_2,s_1) \mathbf{H}(s_1) \\ &= I - \int_{t_1}^{t_2} ds_1 \mathbf{H}(s_1) + \int_{t_1}^{t_2} ds_1 \int_{s_1}^{t_2} ds_2\, R(t_2,s_2)\mathbf{H}(s_2)\mathbf{H}(s_1) \\ &= \cdots = \sum_{j=0}^{\infty} (-1)^j \int_{t_1 \le s_1 \le s_2 \cdots \le s_j \le t_2} ds_1 \cdots ds_j\, \mathbf{H}(s_n) \cdots \mathbf{H}(s_1) \\ &= T e^{-\int_{t_1}^{t_2} \mathbf{H}(s) ds} \,, \end{aligned} \tag{II.21}$$

leading to (II.19).

Lemma II.3. *The interaction $\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s))$ satisfies*

$$\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s)) e^{a^*(e^{-t\omega} f)} = e^{a^*(e^{-t\omega} f)} \mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(f_t + g_s)) \,. \tag{II.22}$$

The corresponding relation for an annihilation operator is

$$e^{a(e^{-t\omega} h)} \mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s)) = \mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(h_t + g_s))\, e^{a(e^{-t\omega} h)} \,. \tag{II.23}$$

Proof. Denote $\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s))$ by $\mathbf{H}_I(s)$. Then

$$\left[\mathbf{H}_I(s), e^{a^*(e^{-t\omega} f)} \right] = e^{a^*(e^{-t\omega} f)} \left(e^{-a^*(e^{-t\omega} f)} \mathbf{H}_I(s) e^{a^*(e^{-t\omega} f)} - \mathbf{H}_I(s) \right) \,. \tag{II.24}$$

But

$$\left[-a^*(e^{-t\omega}f), \mathbf{H}_I(s)\right] = -\mathrm{Ad}_{a^*(e^{-t\omega}f)}\left(\mathbf{H}_I(s)\right) = H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s) * \iota\,(f_t)) \;. \tag{II.25}$$

Expanding the exponential $e^{-a^*}\mathbf{H}_I(s)e^{a^*}$ in (II.24) as a series in $(-\mathrm{Ad}_{a^*})^j$, one obtains

$$\begin{aligned}
\left[\mathbf{H}_I(s), e^{a^*(e^{-t\omega}f)}\right] &= e^{a^*(e^{-t\omega}f)} \sum_{j=1}^{N} \frac{1}{j!}\left(-\mathrm{Ad}_{a^*(e^{-t\omega}f)}\right)^j \left(\mathbf{H}_I(s)\right) \\
&= e^{a^*(e^{-t\omega}f)} \sum_{j=1}^{\infty} \frac{1}{j!} H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s)\,(*\iota\,(f_t))^j) \\
&= e^{a^*(e^{-t\omega}f)} \left(H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s) * \mathbf{\Gamma}(f_t)) - H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s))\right) \\
&= e^{a^*(e^{-t\omega}f)} \left(H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s + f_t)) - H_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_s))\right) \;,
\end{aligned} \tag{II.26}$$

where we use (II.10). Thus we obtain (II.22) as claimed. A similar argument establishes the corresponding relation (II.23).

Proof of Theorem II.1. Let us begin by establishing the case $h = 0$, namely

$$\left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s))ds}\right) e^{a^*(f)} = e^{a^*(e^{-\beta\omega}f)} \left(Te^{-\int_0^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(f_s+g_s))ds}\right) \;. \tag{II.27}$$

Consider the function

$$G(s') = R(\beta, s')e^{a^*(e^{-s'\omega}f)}S(s', 0) \;, \tag{II.28}$$

where

$$R(\beta, s') = \left(Te^{-\int_{s'}^\beta \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s))ds}\right) \;, \qquad \text{and } S(s', 0) = \left(Te^{-\int_0^{s'} \mathbf{H}(\mathcal{P},\lambda\mathbf{\Gamma}(g_s+f_s))ds}\right) \;. \tag{II.29}$$

The left and right sides of (II.15) equal respectively $G(0)$ and $G(\beta)$. We compute the derivative of $G(s)$ and show that it vanishes, proving (II.15). In fact using Proposition II.2, along with the relation

$$\begin{aligned}
\frac{d}{ds}e^{a^*(e^{-s\omega}f)} &= -a^*(\omega e^{-s\omega}f)e^{a^*(e^{-s\omega}f)} = -\left[H_0, a^*(e^{-s\omega}f)\right]e^{a^*(e^{-s\omega}f)} \\
&= -\left[H_0, e^{a^*(e^{-s\omega}f)}\right] \;,
\end{aligned} \tag{II.30}$$

we find that

$$\begin{aligned}
&\frac{d}{ds}G(s') \\
&= R(\beta, s')\left(\mathbf{H}(\mathcal{P}, \lambda\mathbf{\Gamma}(g_{s'}))e^{a^*(e^{-s'\omega}f)} - \left(\frac{d}{ds'}e^{a^*(e^{-s'\omega}f)}\right) - e^{a^*(e^{-s'\omega}f)}\mathbf{H}(\mathcal{P}, \lambda\mathbf{\Gamma}(g_{s'} + f_{s'}))\right)S(s', 0) \\
&= R(\beta, s')\left(\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_{s'}))e^{a^*(e^{-s'\omega}f)} - e^{a^*(e^{-s'\omega}f)}\mathbf{H}_I(\mathcal{P}, \lambda\mathbf{\Gamma}(g_{s'} + f_{s'}))\right)S(s', 0) \;.
\end{aligned} \tag{II.31}$$

Using Lemma II.3, we infer that $dG(s)/ds = 0$ as claimed and (II.27) holds.

Next consider the case $f = 0$, which we analyze by taking the adjoint of the case established above, but with $\bar{g}$ in place of g and $\bar{h}$ in place of f. This gives

$$e^{a(h)} \left(A e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(g_s)) ds} \right) = \left(A e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(h_s + g_s)) ds} \right) e^{a(e^{-\beta\omega} h)} \, , \tag{II.32}$$

where A denotes anti-time ordering. Replacing s by $\beta - s$ in the integrands is equivalent to the replacement of anti-time-ordering by time-ordering. Therefore,

$$e^{a(h)} \left(T e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(g_{\beta-s})) ds} \right) = \left(T e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(h_{\beta-s} + g_{\beta-s})) ds} \right) e^{a(e^{-\beta\omega} h)} \, . \tag{II.33}$$

In order to combine the two expressions, replace $g_{\beta-s}$ by g_s to yield,

$$e^{a(h)} \left(T e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(g_s)) ds} \right) = \left(T e^{-\int_0^\beta \mathbf{H}(\mathcal{P}, \lambda \mathbf{\Gamma}(h_{\beta-s} + g_s)) ds} \right) e^{a(e^{-\beta\omega} h)} \, . \tag{II.34}$$

Multiply this identity on the right by $e^{a^*(f)}$. Then move this exponential to the left in the right-hand term: use the canonical commutation relations to commute $e^{a^*(f)}$ past $e^{a(e^{-\beta\omega} h)}$. Then apply the exchange identity (II.27) that was already proved. This yields (II.15) and completes the proof of the theorem.

References

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