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Quantum Invariants*

Arthur Jaffe

Harvard University, Cambridge, MA 02138, USA

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Abstract: In earlier work, we derived an expression for a partition function $\mathbf{\mathfrak{J}}^{(\lambda)}$, and gave a set of analytic hypotheses under which $\mathbf{\mathfrak{J}}^{(\lambda)}$ does not depend on a parameter λ . The proof that $\mathbf{\mathfrak{J}}^{(\lambda)}$ is invariant involved entire cyclic cohomology and *K*-theory. Here we give a direct proof that $\frac{d}{d\lambda}\mathbf{\mathfrak{J}}^{(\lambda)} = 0$. The considerations apply to non-commutative geometry, to super-symmetric quantum theory, to string theory, and to generalizations of these theories to underlying quantum spaces.

1. Introduction

In [QHA] we studied a class of geometric index invariants, in (non-commutative) differential geometry [C1, C2, JLO]. These invariants arise from pairing a cocycle $\tau^{\lambda,g}$ in (equivariant) entire cyclic cohomology, with an operator square-root of unity *a*. In [JLO] we discovered a representation τ^{JLO} of a cocycle defined in terms of a given self-adjoint operator $Q(\lambda)$, yielding $\mathbf{3}^{\lambda}(a,g) = \mathbf{3}^{Q(\lambda)}(a,g) = \langle \tau^{\lambda,g}, a \rangle$. In [QHA] we study a modification of this pairing, and extend the class of perturbations of $Q(\lambda)$ under which the invariants are stable, formulating a sufficient condition of *fractional differentiability*. The index studied in [QHA] has the numerical value

$$\mathfrak{Z}^{\mathcal{Q}(\lambda)}(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathcal{H}}(\gamma U(g)ae^{-\mathcal{Q}(\lambda)^2 + itd_{\lambda}a})e^{-t^2}dt, \qquad (1.1)$$

where $Q(\lambda)$ acts on a Hilbert space \mathcal{H} , and the differential $da = d_{\lambda}a$ is defined by $da = [Q(\lambda), a]$. We assume that Q is odd with respect to the \mathbb{Z}_2 -grading γ , while a is even. We also assume that g is an element of a group G of symmetries of Q and of a. The invariant $\mathfrak{Z}^Q(a, g)$ does not necessarily take integer values, but it is integer in case g equals the identity.

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In this note we present an elementary analysis of the invariance of (1.1). Rather than relating **3** to cohomology or *K*-theory, we study the end result of that analysis. We ask: can one show directly that $\mathbf{\mathfrak{Z}}^{Q(\lambda)}(a, g)$ is constant? We answer this question affirmatively, by introducing an auxiliary Hilbert space $\hat{\mathcal{H}}$ that is a skew tensor-product of \mathcal{H} with a finite dimensional space. We obtain a representation for **3** as an expectation $\mathbf{\mathfrak{J}}(\lambda, a, g)$ on $\hat{\mathcal{H}}$. Therefore we replace the study of $Q(\lambda)$ acting on \mathcal{H} by the study of $q(\lambda, a) = Q(\lambda) + \eta a$ acting on $\hat{\mathcal{H}}$. Using the identity $\mathbf{\mathfrak{Z}}^{Q(\lambda)}(a, g) = \mathbf{\mathfrak{J}}(\lambda, a, g)$, it becomes an elementary calculation to establish that $\frac{d}{d\lambda}\mathbf{\mathfrak{Z}}^{Q(\lambda)}(a, g) = 0$.

The Hilbert space $\hat{\mathcal{H}}$ differs from \mathcal{H} by also containing the additional independent fermionic coordinate η chosen so that $\eta^2 = I$, $\eta a = a\eta$, and $\eta Q(\lambda) + Q(\lambda)\eta = 0$. We interpret ηa as a connection associated with the translation in the auxiliary direction t, paired with the fermionic coordinate η .

We present the algebraic aspects of the proof, without the analytic details. These analytic details remain absolutely crucial, and without them we would be in the position to show that all invariants for a given *a* agree! For example, if any two odd Q and \tilde{Q} both commute with the symmetry group U(g), then we could take $Q(\lambda) = \lambda Q + (1-\lambda)\tilde{Q}$ and attempt to interpolate between them for $0 \le \lambda \le 1$. However the analytic assumptions we require differ little from those in [QHA], and in fact they comprise a major portion of that work. First we define the regularity of $Q(\lambda)$ with respect to λ , and secondly we investigate the regularity of *a* with respect to commutation with $Q(\lambda)$. We call the latter the *fractional differentiability* properties of *a*. The conditions in [QHA] are convenient in many examples, and we summarize our analytic hypotheses in Sect. 11. Under these regularity conditions, $\mathbf{3}^{Q(\lambda)}(a, g)$ is once-differentiable in λ . Furthermore, the resulting λ -derivative of $\mathbf{3}$ equals the expression that we would obtain by interchanging the order of differentiating and the order of taking traces or integrals in the definition of $\mathbf{3}$.

In Sect. 10 we consider a different but related case with two independent differentials Q_1 and Q_2 . While we assume that Q_1 is invariant under the symmetry group G, we do not assume the invariance of Q_2 . We replace this assumption by the two assumptions, namely both that Q_2^2 is invariant, and that $Q_1^2 - Q_2^2$ commutes with all observables. We show in this case that an expectation (10.4) has a representation similar to (1.1) and also is an invariant with respect to λ .

2. The Supercharge

Our basic framework involves an odd, self adjoint operator Q on a \mathbb{Z}_2 -graded Hilbert space \mathcal{H} . This means that we have a self-adjoint operator γ on \mathcal{H} for which $\gamma^2 = I$. Thus \mathcal{H} splits into the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of eigenspaces of γ . The statement that Q is odd means $Q\gamma + \gamma Q = 0$. In terms of the direct sum decomposition,

$$Q = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$
 (2.1)

The operator Q is the supercharge¹ and its square

$$H = Q^{2} = \begin{pmatrix} Q_{+}^{*}Q_{+} & 0\\ 0 & Q_{+}Q_{+}^{*} \end{pmatrix}$$
(2.2)

¹ We are not concerned with the basic structure of \mathcal{H} or Q, aside from the possibility to perform the construction in §V.

will be referred to as the Hamiltonian. We let $x^{\gamma} = \gamma x \gamma$ denote the action of γ on operators. We say that the operator *x* is even (bosonic) if $x = x^{\gamma}$ and odd (fermionic) if $x^{\gamma} = -x$. We define the graded differential

$$dx = Qx - x^{\gamma}Q. \tag{2.3}$$

We suppose that there is a compact Lie group G with a continuous unitary representation U(g) on \mathcal{H} such that

$$U(g)\gamma = \gamma U(g), \text{ and } U(g)Q = QU(g).$$
 (2.4)

Denote the action of U(g) on the operator x by

$$x \to x^g = U(g)xU(g)^{-1}.$$
 (2.5)

3. The Observables

Consider an algebra of bounded operators \mathfrak{A} on \mathcal{H} such that each $a \in \mathfrak{A}$ is even and *g*-invariant. In other words, each $a \in \mathfrak{A}$ commutes with γ and with U(g) for all $g \in G$. Also consider Mat_n(\mathfrak{A}), the set of $n \times n$ matrices with matrix elements in \mathfrak{A} . If $a \in Mat_n(\mathfrak{A})$, then $a = \{a_{ij}\}$, where $a_{ij} \in \mathfrak{A}$. Use the shorthand $ab \in Mat_n(\mathfrak{A})$ to denote the matrix with entries $(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \in \mathfrak{A}$. Define the differential of a element $a \in \mathfrak{A}$ by

$$da = Qa - aQ = [Q, a].$$
 (3.1)

This is always densely defined as a quadratic form on $\mathcal{H} \times \mathcal{H}$. We make precise the boundedness properties of this quadratic form in Sect. 11. We use *a* to denote an element of the algebra **21** and *x* to denote a linear operator or a bilinear form acting on \mathcal{H} . In the latter case, we assume that the domain of *x* includes $C^{\infty}(Q(\lambda)) \times C^{\infty}(Q(\lambda))$.

4. The Invariant $\mathfrak{Z}^{Q(\lambda)}(a,g)$

In [QHA] we gave a simple formula for an invariant. Let $Q(\lambda)$ depend on a real parameter λ . We denote the graded commutator (2.3) of $Q(\lambda)$ with *x* by

$$d_{\lambda}x = Q(\lambda)x - x^{\gamma}Q(\lambda), \qquad (4.1)$$

that reduces to $d_{\lambda}a = [Q(\lambda), a]$ for $a \in \mathfrak{A}$. For $a \in \mathfrak{A}$, the invariant is (1.1). More generally, we let $a \in Mat_n(\mathfrak{A})$. In this case

$$\mathfrak{Z}^{\mathcal{Q}(\lambda)}(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathcal{H}\otimes\mathbb{C}^{n^2}} \left(\gamma U(g) a e^{-\mathcal{Q}(\lambda)^2 + itd_{\lambda}a} \right) e^{-t^2} dt , \qquad (4.2)$$

where γ , U(g), $Q(\lambda)^2$ act in Mat_n(\mathfrak{A}) as diagonal $n \times n$ matrices of the form $\gamma \otimes I$, $U(g) \otimes I$, etc.

Theorem 1. For $a \in Mat_n(\mathfrak{A})$, assume $a^2 = I$. Furthermore assume that $Q = Q(\lambda)$ and $d_{\lambda}a = [Q(\lambda), a]$ satisfy the regularity hypotheses given in Sect. 11. Then $\mathfrak{Z}^{Q(\lambda)}(a, g)$ is independent of λ .

The main point of this paper is to present an elementary proof of Theorem 1.

5. The Extended Supercharge q

In order to exhibit our proof, we introduce a new Hilbert space $\hat{\mathcal{H}}$ on which the operators Q, γ, \mathfrak{A} , and U(g) also act. In addition, on $\hat{\mathcal{H}}$ there are two additional self adjoint operators η and J, both of which have square one,

$$\eta^2 = J^2 = I. (5.1)$$

Furthermore, we assume that η commutes with γ , with all elements of \mathfrak{A} , and with the representation U(g). In other words,

$$[\eta, x] = [J, x] = 0 \text{ for } x = \gamma, a \in \mathfrak{A}, \text{ or } U(g).$$

$$(5.2)$$

Also we assume that J commutes with Q, but that η anticommutes with J and with Q,

$$\eta J + J\eta = \eta Q + Q\eta = [J, Q] = 0.$$
(5.3)

Let $\Gamma = \gamma J$ denote a \mathbb{Z}_2 -grading on $\hat{\mathcal{H}}$. We now let *x* denote a linear operator or bilinear form acting on $\hat{\mathcal{H}}$ (in the latter case, with domain including $C^{\infty}(Q(\lambda)) \times C^{\infty}(Q(\lambda))$

$$x^{\Gamma} = \Gamma x \Gamma. \tag{5.4}$$

The operator η is our auxiliary fermionic coordinate, and $J = (-I)^{N_{\eta}}$ is the corresponding \mathbb{Z}_2 grading.²

Given $a \in \mathfrak{A}$, define the extended supercharge $q = q(\lambda, a)$ by

$$q = q(\lambda, a) = Q(\lambda) + \eta a, \tag{5.5}$$

and also let

$$h = h(\lambda, a) = q(\lambda, a)^2 = Q(\lambda)^2 + a^2 - \eta d_\lambda a.$$
 (5.6)

Note that

$$q^{\Gamma} = -q, \text{ and } h^{\Gamma} = h.$$
 (5.7)

We use the notation d_q to denote the Γ -graded commutator on $\hat{\mathcal{H}}$,

$$d_q x = q x - x^{\Gamma} q. \tag{5.8}$$

If we need to emphasize the dependence of q on λ or a, then we write $d_{q(\lambda,a)}x$. We continue to reserve d or d_{λ} to denote the γ -graded commutator (4.1).

² Suppose that $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ is a tensor product of bosonic and fermionic Fock spaces, that Q is linear in fermionic creation or annihilation operators, and that $\gamma = (-I)^{N_f}$. This would be standard in the physics of supersymmetry. Suppose in addition that $\eta = b + b^*$ denotes one fermionic degree of freedom independent of those in \mathcal{H}_f and acting on the two-dimensional space \mathcal{H}_η . Then take $\hat{\mathcal{H}} = \mathcal{H}_b \otimes (\mathcal{H}_f \wedge \mathcal{H}_\eta)$ and $J = (-I)^{b^*b}$, with Q, γ, a , and U(g) acting on $\hat{\mathcal{H}}$ in the natural way. This gives a realization of (V.1–3) on $\hat{\mathcal{H}}$.

6. Heat Kernel Regularization on $\hat{\mathcal{H}}$

Let us introduce the heat kernel regularizations \hat{X}_n of X_n on $\hat{\mathcal{H}}$. Let $X_n = \{x_0, \ldots, x_n\}$ denote an ordered set of (n + 1) linear operators x_j acting on $\hat{\mathcal{H}}$. We call the x_j vertices and X_n a set of vertices. Choose $a \in \mathfrak{A}$ and let $q(\lambda, a) = Q(\lambda) + \eta a$, and $h = h(\lambda, a) = q(\lambda, a)^2$. Define the heat kernel regularization $\hat{X}_n(\lambda, a) = \{x_0, \ldots, x_n\}^{\wedge}(\lambda, a)$ by

$$\hat{X}_n(\lambda, a) = \int_{s_j > 0} x_0 e^{-s_0 h} x_1 e^{-s_1 h} \cdots x_n e^{-s_n h} \delta(1 - s_0 - \dots - s_n) ds_0 \cdots ds_n.$$
(6.1)

Note that if T is any operator on $\hat{\mathcal{H}}$ that commutes with $h = q^2$, then

$$\{x_0, \dots, x_j T, x_{j+1}, \dots, x_n\}^{\wedge}(\lambda, a) = \{x_0, \dots, x_j, T x_{j+1}, \dots, x_n\}^{\wedge}(\lambda, a).$$
(6.2)

Furthermore $T = J\eta$ anti-commutes with $q(\lambda, a)$ and commutes with $h(\lambda, a)$ for all a.

Proposition 2 (Vertex Insertion). Let $X_n = \{x_0, ..., x_n\}$ denote a set of vertices possibly depending on λ . Then with the notation $\dot{Q} = \partial Q(\lambda)/\partial \lambda$, we have

$$\frac{\partial}{\partial\lambda} \{x_0, \dots, x_n\}^{\wedge}(\lambda, a) = -\sum_{j=0}^n \{x_0, \dots, x_j, d_q \dot{Q}, x_{j+1}, \dots, x_n\}^{\wedge}(\lambda, a) + \sum_{j=0}^n \{x_0, \dots, \frac{\partial x_j}{\partial\lambda}, \dots, x_n\}^{\wedge}(\lambda, a).$$
(6.3)

Here

$$d_q \dot{Q} = d_{q(\lambda,a)} \dot{Q} = d_\lambda \dot{Q} + \eta[a, \dot{Q}].$$
(6.4)

Proof. By differentiating \hat{X}_n defined in (6.1), we obtain two types of terms. Differentiating the x_j 's gives the second sum in (6.3). (This sum is absent if the x_j 's are λ -independent.) The other terms arise from differentiating the heat kernels. We use the identity

$$\frac{\partial}{\partial\lambda}e^{-sh} = -\int_0^s e^{-uh} \frac{\partial h}{\partial\lambda} e^{-(s-u)h} du, \tag{6.5}$$

that holds under suitable regularity hypotheses, see for example Proposition VII.10 of [QHA]. Here

$$\frac{\partial h}{\partial \lambda} = \frac{\partial}{\partial \lambda} (q^2) = q \frac{\partial q}{\partial \lambda} + \frac{\partial q}{\partial \lambda} q = d_q \left(\frac{\partial q}{\partial \lambda} \right) = d_q \dot{Q}.$$

Explicitly

$$d_q \dot{Q} = (Q + \eta a) \dot{Q} + \dot{Q}(Q + \eta a) = d_\lambda \dot{Q} + \eta [a, \dot{Q}].$$

Inserted back into the definition of \hat{X}_n , we observe that the differentiation of the heat kernel between vertex j and vertex j + 1 produces one new $-d_q \dot{Q}$ vertex at position j + 1. This completes the proof of (6.3). \Box

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Define the action of the grading Γ on sets of vertices X_n by

$$X_n \to X_n^{\Gamma} = \{x_0^{\Gamma}, x_1^{\Gamma}, \dots, x_n^{\Gamma}\}.$$
(6.6)

Since $q^2 = (q^{\Gamma})^2$, the regularization $X_n \to \hat{X}_n$ commutes with the action of Γ , namely

$$\left(\hat{X}_n(\lambda, a)\right)^{\Gamma} = \left(X_n^{\Gamma}\right)^{\wedge}(\lambda, a).$$
(6.7)

It is also convenient to write explicitly the expression for the differential $d_q \hat{X}_n$,

$$d_q \hat{X}_n(\lambda, a) = q \hat{X}_n(\lambda, a) - \hat{X}_n(\lambda, a)^{\Gamma} q$$

= {qx₀, x₁,..., x_n}^(\lambda, a) - {x₀^{\Gamma}, ..., x_n^{\Gamma}q}^(\lambda, a). (6.8)

In particular, we infer that

$$d_q \hat{X}_n(\lambda, a) = \sum_{j=0}^n \{x_0^{\Gamma}, x_1^{\Gamma}, \dots, x_{j-1}^{\Gamma}, d_q x_j, \dots, x_n\}^{\wedge}(\lambda, a).$$
(6.9)

One other identity we mention is

Proposition 3 (Combination Identity). The heat kernel regularizations satisfy

$$\{x_0, x_1, \dots, x_n\}^{\wedge}(\lambda, a) = \sum_{j=0}^n \{x_0, x_1, \dots, x_j, I, x_{j+1}, \dots, x_n\}^{\wedge}(\lambda, a).$$
(6.10)

Proof. The j^{th} term on the right side of (6.9) is

$$\{x_0, \dots, x_j, I, x_{j+1}, \dots, x_n\}^{\wedge}(\lambda, a) = \int_{s_j > 0} x_0 e^{-s_0 h} \cdots x_j e^{-(s_j + s_{j+1})h} \\ \cdots x_n e^{-s_{n+1}h} \delta(1 - s_0 - \dots - s_{n+1}) ds_0 \cdots ds_{n+1}.$$
(6.11)

Change the *s*-integration variables to $s'_0 = s_0, s'_1 = s_1, \ldots, s'_j = s_j + s_{j+1}, s'_{j+1} = s_{j+2}, \ldots, s'_n = s_{n+1}$, and $s'_{n+1} = s_j$. This change has Jacobian 1, and the resulting integrand has the form of the integrand for $\{x_0, \ldots, x_n\}^{\wedge}$ with variables s'_0, \ldots, s'_n , namely

$$\int_{s'_0, s'_1, \dots, s'_n > 0} ds'_0 \cdots ds'_n \left(\int ds'_{n+1} x_0 e^{-s'_0 h} \cdots x_n e^{-s'_n h} \delta(1 - s'_0 - \dots - s'_n) \right),$$
(6.12)

with the integrand depending on the variable s'_{n+1} only through the restriction of the range of the s'_{n+1} integral. The original domain of integration restricts s'_{n+1} to the range $0 \le s'_{n+1} \le s'_j$, so the dependence of the integrand on s'_{n+1} is the characteristic function of the interval $[0, s'_j]$. Thus performing the s'_{n+1} integration produces a factor s'_j in the s'_0, \ldots, s'_n -integrand. Add the similar results for $0 \le j \le n$ to give the factor $s'_0 + s'_1 + \cdots + s'_n$. But the delta function in (6.12) restricts this sum to be 1, so the integral of the sum is exactly $\{x_0, \ldots, x_n\}^{\wedge}(\lambda, a)$. \Box

7. Expectations on $\hat{\mathcal{H}}$

Let $a \in \mathfrak{A}$ satisfy $a^2 = I$, and let $\hat{X}_n = \hat{X}_n(\lambda, a)$ denote the heat kernel regularization of X_n . We define the expectation

$$\left\langle \left\langle \hat{X}_{n} \right\rangle \right\rangle_{\lambda,a,g} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \operatorname{Tr}_{\hat{\mathcal{H}}} \left(\Gamma U(g) \hat{X}_{n}(\lambda, ta) \right) dt.$$
 (7.1)

Here we choose $a^2 = I$ to ensure that the t^2 term in h provides a gaussian convergence factor to the *t*-integral. This integral represents averaging over *a*'s whose squares are multiples of the identity.

These expectations can be considered as (n + 1)-multilinear expectations on sets X_n of vertices. We sometimes suppress the λ - or *a*- or *g*-dependence of the expectations, or the *n*-dependence of sets of vertices. Furthermore, where confusion does not occur we omit the $^{\wedge}$ that we use to distinguish a set of vertices X from the heat kernel regularization of the set. Thus at various times we denote $\langle \langle \hat{X}_n \rangle \rangle_{a,g}$ by $\langle \langle X \rangle \rangle$, or when we wish to clarify the dependence on *n*, *a*, or *g* with some subset of these indices, or even as one of the following:

$$\left\langle \left(\hat{X}_n \right) \right\rangle_{\lambda,a,g} = \langle \langle X \rangle \rangle = \langle \langle X \rangle \rangle_n = \langle \langle X \rangle \rangle_{n,a} = \langle \langle X \rangle \rangle_{n,a,g} , \qquad (7.2)$$

etc.

Proposition 4. With the above notation, we have the identities

 $(\Gamma$ -invariance)

$$\langle\langle X \rangle\rangle_n = \langle\!\langle X^\Gamma \rangle\!\rangle_n,$$
 (7.3)

(dif

$$\left\langle\!\left\langle d_{q}X\right\rangle\!\right\rangle_{n} = \sum_{j=0}^{n} \left\langle\!\left\langle\!\left\{x_{0}^{\Gamma}, x_{1}^{\Gamma}, \dots, x_{j-1}^{\Gamma}, d_{q}x_{j}, \dots, x_{n}\right\}\!\right\rangle\!\right\rangle,\tag{7.4}$$

(cyclic

$$\langle\langle\{x_0, x_1, \dots, x_n\}\rangle\rangle = \left\langle\!\left\langle\!\left\{x_n^{g^{-1}\Gamma}, x_1, x_2, \dots, x_{n-1}\right\}\right\rangle\!\right\rangle,\tag{7.5}$$

and

(combination identity)

$$\langle \langle \{x_0, x_1, \dots, x_n\} \rangle \rangle_n = \sum_{j=0}^n \langle \langle \{x_0, x_1, \dots, x_j, I, x_{j+1}, \dots, x_n\} \rangle \rangle_{n+1}.$$
 (7.6)

Also, in case $Q = Q^g$ and $a = a^g$, then $q = q^g$ and we have

(infinitesimal invariance)

$$\langle\!\langle d_{q(\lambda,ta)}X\rangle\!\rangle = 0.$$
 (7.7)

Proof. The symmetry (7.3) is a consequence of the fact that $\Gamma^2 = I$, and Γ commutes with U(g) and with q^2 . The expectation of (6.9) completes the proof of (7.4). The proof of (7.5) involves cyclicity of the trace. The identity (7.6) is the expectation of (6.10). To establish (7.7), note that every \hat{X}_n can be decomposed uniquely as $\hat{X}_n = \hat{X}_n^+ + \hat{X}_n^-$, where $(\hat{X}_n^{\pm})^{\Gamma} = \pm \hat{X}_n^{\pm}$. The symmetry (7.3) ensures that $\langle \langle d_{q(\lambda,ta)} X_n^+ \rangle \rangle = 0$. On the other hand, $q^{\Gamma} = -q$, together with cyclicity of the trace and $q^g = q$ ensures that

$$\begin{split} \left\langle \left\langle d_{q(\lambda,ta)} X_n^{-} \right\rangle \right\rangle &= \left\langle \left\langle q(\lambda,ta) X_n^{-} \right\rangle \right\rangle + \left\langle \left\langle X_n^{-} q(\lambda,ta) \right\rangle \right\rangle \\ &= \left\langle \left\langle q(\lambda,ta) X_n^{-} \right\rangle \right\rangle + \left\langle \left\langle q(\lambda,ta)^{g^{-1}\Gamma} X_n^{-} \right\rangle \right\rangle = 0. \quad \Box \end{split}$$

Except in (7.7), we have implicitly assumed that the vertices x_j in X_n are *t*-independent. In case that X_n has one factor linear in *t*, the heat kernel regularizations of the following agree:

$$\{tx_0, x_1, \dots, x_n\}^{\wedge} (\lambda, ta) = \{x_0, x_1, \dots, tx_j, \dots, x_n\}^{\wedge} (\lambda, ta),$$
(7.8)

for any j = 0, 1, ..., n. We then obtain an interesting relation for expectations,

Proposition 5 (Integration by parts). Let $a^2 = I$. Then

$$\langle\langle\{tx_0, x_1, \dots, x_n\}\rangle\rangle_n = \sum_{j=0}^n \frac{1}{2} \langle\langle\{x_0, \dots, x_j, \eta d_{\lambda} a, x_{j+1}, \dots, x_n\}\rangle\rangle_{n+1}.$$
(7.9)

Proof. In order to establish (7.9), we collect together the terms $\exp(-s_jt^2)$ that occur in $\{x_0, \ldots, x_n\}^{\wedge}$ (*ta*). Since the integrand for the heat kernel regularization has a δ -function restricting the variables s_j to satisfy $s_0 + \cdots + s_n = 1$, we obtain the factor $\exp(-t^2)$. Write

$$te^{-t^2} = -\frac{1}{2}\frac{d}{dt}\left(e^{-t^2}\right)$$

and integrate by parts in t. The resulting derivative involves the t-derivative of each heat kernel exp $-(s_i q(\lambda, ta)^2)$ with the quadratic term in t removed from q^2 . Note that

$$e^{-st^2} \frac{d}{dt} e^{-s(q^2 - t^2)} = -\int_0^s e^{-uq^2} \left(\frac{d}{dt}(q^2 - t^2)\right) e^{-(s-u)q^2} du$$
$$= \int_0^s e^{-uq^2} \eta d_\lambda a e^{-(s-u)q^2} du.$$

Here we use (5.6) with *ta* replacing *a* and with $a^2 = I$ in order to evaluate the *t* derivative of $q^2 - t^2$. Thus each derivative introduces a new vertex equal to $\frac{1}{2}\eta d_{\lambda}a$, and the proof of (7.9) is complete. \Box

8. The Functional $\mathfrak{J}(\lambda, a, g)$

Let us consider a single vertex and $X_0 = x_0 = Ja$, where $a \in \mathfrak{A}$, and its expectation

$$\mathfrak{J}(\lambda, a, g) = \langle \langle Ja \rangle \rangle. \tag{8.1}$$

Explicitly

$$\mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \operatorname{Tr}_{\hat{\mathcal{H}}} \left(\gamma U(g) a e^{-q(\lambda, ta)^2} \right) dt.$$
(8.2)

This functional allows us to recover the functional \mathfrak{Z} .

Theorem 6. Let a satisfy $a^2 = I$. Then

$$\mathfrak{J}(\lambda, a, g) = \mathfrak{Z}^{\mathcal{Q}(\lambda)}(a, g). \tag{8.3}$$

Proof. Let $h = h_0 - t\eta da$, where $h_0 = Q(\lambda)^2 + t^2$. The Hille–Phillips perturbation theory for semi-groups, see Theorem 13.4.1 of [HP], can be written

$$e^{-q(\lambda,ta)^{2}} = e^{-h_{0}+t\eta da}$$

= $e^{-h_{0}} + \sum_{n=1}^{\infty} t^{n} \int_{s_{j}>0} e^{-s_{0}h_{0}} \eta d_{\lambda} a e^{-s_{1}h_{0}} \eta d_{\lambda} a$
 $\cdots \eta d_{\lambda} a e^{-s_{n}h_{0}} \delta(1-s_{0}-s_{1}-\cdots-s_{n}) ds_{0} ds_{1}\cdots ds_{n}.$ (8.4)

In the *n*th term we collect all factors of η on the left. Note that η commutes with *a* and h_0 , and it anti-commutes with $d_{\lambda}a$. Therefore the result of collecting the factors of η on the left is $\eta^n (-1)^{n(n-1)/2}$. If *n* is odd, then $\eta^n = \eta$ and $\operatorname{Tr}_{\mathcal{H}_{\eta}}(\eta) = 0$. Thus only even *n* terms contribute to (8.2). For even *n*, $\eta^n (-1)^{n(n-1)/2} = (-1)^{n/2}I$ and $\operatorname{Tr}_{\mathcal{H}_{\eta}}(I) = 2$. Thus (8.2) becomes

$$\mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty} (-t^2)^n \langle \{a, d_\lambda a, \dots, d_\lambda a\} \rangle_{2n} e^{-t^2}, \qquad (8.5)$$

where we use expectations $\langle \rangle_n$ on \mathcal{H} similar to $\langle \langle \rangle \rangle_n$ on $\hat{\mathcal{H}}$ (but without the *t*-integration) and defined by

$$\langle \{x_0, \dots, x_n\} \rangle_n = \int_{s_j > 0} \operatorname{Tr}_{\mathcal{H}} \left(\gamma U(g) x_0 e^{-s_0 \mathcal{Q}(\lambda)^2} \cdots x_n e^{-s_n \mathcal{Q}(\lambda)^2} \right)$$
$$\cdot \delta(1 - s_0 - s_1 - \dots - s_n) ds_0 ds_1 \cdots ds_n.$$
(8.6)

But using the Hille–Phillips formula once again, (8.6) is just

$$\frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} dt \operatorname{Tr}_{\mathcal{H}} \left(\gamma U(g) a e^{-\mathcal{Q}(\lambda)^2 + itd_{\lambda}a} \right) e^{-t^2} = \mathfrak{Z}^{\mathcal{Q}(\lambda)}(a,g).$$
(8.7)

(Here we use the symmetry of (8.7) under γ to justify vanishing of terms involving odd powers of $d_{\lambda}a$.) Thus we can prove that $\mathfrak{J}^{Q(\lambda)}(a, g)$ is independent of λ by showing that $\mathfrak{J}(\lambda, a, g)$ is constant in λ . \Box

9. $(\mathfrak{J}(\lambda, a, g))$ Does Not Depend on λ

We now prove Theorem 1. Calculate $\partial \mathfrak{J} / \partial \lambda$ using (6.3), in the simple case of one vertex independent of λ . Thus

$$\frac{\partial}{\partial\lambda}\mathfrak{J}(\lambda, a, g) = \frac{\partial}{\partial\lambda} \left\langle \left\langle Ja \right\rangle \right\rangle = -\left\langle \left\langle \left\{ Ja, d_{q(\lambda, ta)}\dot{Q} \right\} \right\rangle \right\rangle. \tag{9.1}$$

Using the identity (7.7) in the form

$$0 = \langle\!\langle d_{q(\lambda,ta)} \left\{ Ja, \dot{Q} \right\}\!\rangle\!\rangle = \langle\!\langle \left\{ d_{q(\lambda,ta)}(Ja), \dot{Q} \right\}\!\rangle\!\rangle + \langle\!\langle \left\{ Ja, d_{q(\lambda,ta)}\dot{Q} \right\}\!\rangle\!\rangle , \qquad (9.2)$$

we have

$$\frac{\partial}{\partial\lambda}\mathfrak{J}(\lambda, a, g) = \left\langle \left| \left\{ d_{q(\lambda, ta)}(Ja), \dot{Q} \right\} \right\rangle \right\rangle.$$
(9.3)

It is at this point that we have used $q^g = q$, namely the invariance of both Q and a under U(g). To evaluate (9.3), note that

$$d_{q(\lambda,ta)}(Ja) = [q(\lambda,ta), Ja] = Jd_{\lambda}a - 2tJ\eta.$$
(9.4)

Here we use the assumption $a^2 = I$. From Proposition V we therefore infer

$$\frac{\partial}{\partial\lambda}\mathfrak{J}(\lambda, a, g) = \langle\!\langle \{Jd_{\lambda}a, \dot{Q}\}\rangle\!\rangle - 2 \langle\!\langle \{tJ\eta, \dot{Q}\}\rangle\!\rangle = \langle\!\langle \{Jd_{\lambda}a, \dot{Q}\}\rangle\!\rangle - \langle\!\langle \{J\eta, \eta d_{\lambda}a, \dot{Q}\}\rangle\!\rangle - \langle\!\langle \{J\eta, \dot{Q}, \eta d_{\lambda}a\}\rangle\!\rangle.$$
(9.5)

Since $J\eta$ commutes with $h = q^2$, and since $J\eta \dot{Q} = -\dot{Q}J\eta$, use (6.2) to establish

$$\langle\!\langle \{J\eta, \eta d_{\lambda}a, \dot{Q}\}\rangle\!\rangle + \langle\!\langle \{J\eta, \dot{Q}, \eta d_{\lambda}a\}\rangle\!\rangle = \langle\!\langle \{I, Jd_{\lambda}a, \dot{Q}\}\rangle\!\rangle - \langle\!\langle \{I, \dot{Q}, Jd_{\lambda}a\}\rangle\!\rangle = \langle\!\langle \{Jd_{\lambda}a, \dot{Q}, I\}\rangle\!\rangle + \langle\!\langle \{Jd_{\lambda}a, I, \dot{Q}\}\rangle\!\rangle.$$
(9.6)

In the last step we also use $\dot{Q}^{\Gamma} = -\dot{Q}$ and the cyclic symmetry (7.5). Hence we can simplify (9.6) to $\langle\!\langle \{Jd_{\lambda}a, \dot{Q}\} \rangle\!\rangle$, by applying the combination identity (7.6). Substituting this back into (9.5), we end up with

$$\frac{\partial}{\partial\lambda}\mathfrak{J}(\lambda, a, g) = \langle\!\langle \{Jd_{\lambda}a, \dot{Q}\}\rangle\!\rangle - \langle\!\langle \{Jd_{\lambda}a, \dot{Q}\}\rangle\!\rangle = 0.$$
(9.7)

Thus $\mathfrak{J}(\lambda, a, g)$ is invariant under change of λ , and the demonstration is complete. \Box

10. Independent Supercharges $Q_i(\lambda)$

Let us generalize our consideration to the case that there are two self-adjoint operators $Q_1 = Q_1(\lambda)$ and $Q_2 = Q_2(\lambda)$ on \mathcal{H} such that

$$Q_1\gamma + \gamma Q_1 = Q_2\gamma + \gamma Q_2 = Q_1Q_2 + Q_2Q_1 = 0.$$
(10.1)

Thus we have two derivatives $d_j a = [Q_j, a]$. We assume that the energy operator on \mathcal{H} is defined by

$$H = H(\lambda) = \frac{1}{2} (Q_1 + Q_2)^2 = \frac{1}{2} \left(Q_1^2 + Q_2^2 \right)$$
(10.2)

and that the operator

$$P = \frac{1}{2} \left(Q_1^2 - Q_2^2 \right) \tag{10.3}$$

has the properties:

i) *P* does not depend on λ .

ii) *P* commutes with Q_1, Q_2 and with each $a \in \mathfrak{A}$.

iii) U(g) commutes with Q_1 and with $H(\lambda)$.

Assumption (i) corresponds to a common situation where P can be interpreted as a "momentum" operator. Then the energy, but not the momentum, is assumed to depend on λ . Assumption (ii) says that Q_1 , Q_2 are translation invariant, and that \mathfrak{A} is a "zero-momentum" or translation-invariant subalgebra. According to assumption (iii), U(g) commutes with Q_2^2 , but U(g) may not commute with Q_2 . Under these hypotheses, and with appropriate regularity assumptions, we proved in [QHA] that for $a = a^g$ and $a^2 = I$,

$$\mathfrak{Z}^{\{Q_j(\lambda)\}}(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{Tr}\left(\gamma U(g) a e^{-H + itd_1 a - t^2}\right) dt \tag{10.4}$$

is independent of λ . In this section we give an alternate proof that (10.4) is constant.

Introduce on $\hat{\mathcal{H}}$ two extended supercharges $q_1 = q_1(\lambda, a) = Q_1 + \eta a$ and $q_2 = Q_2$. With η as before, $\eta Q_1 + Q_1 \eta = \eta Q_2 + Q_2 \eta = 0$. Define

$$h = h(\lambda, ta) = H(\lambda) + t^2 a^2 - t\eta d_1 a.$$
(10.5)

Note that

$$h = q_1(\lambda, ta)^2 - P = Q_1(\lambda)^2 + t^2 a^2 - t\eta d_1 a - P , \qquad (10.6)$$

so we can eliminate $Q_2(\lambda)$ from *h* by introducing the operator *P*, that commutes with *a*, γ , *J*, U(g), η , and $Q_j(\lambda)$.

Thus *P* commutes with all operators that we consider on $\hat{\mathcal{H}}$, so we repeat the constructions of Sects. 5–9. However, we replace $q(\lambda, ta)^2$ in the previous construction with $h(\lambda, ta)$ defined by (10.5). Also we replace $d_q x$ with $d_{q_j} x = q_j x - x^{\Gamma} q_j$. We use the heat kernel $\exp(-sh)$ to define the heat kernel regularization. Then define the expectation $\langle \langle \cdot \rangle \rangle$ by the formula (7.1) with this new $h(\lambda, ta)$. As $q_1 = q_1^g$, therefore we have

$$\left\langle \left\langle d_{q_1(\lambda,ta)}X\right\rangle \right\rangle = 0. \tag{10.7}$$

However it may not be true that $q_2 = q_2^g$, so it may not be true that $\langle \langle d_{q_2(\lambda, ta)}X \rangle \rangle$ vanishes. As before, with $a^2 = I$, define

$$\mathfrak{J}(\lambda, a, g) = \langle \langle Ja \rangle \rangle. \tag{10.8}$$

In this case, we establish as in the proof of Theorem 6 that

$$\mathfrak{J}(\lambda, a, g) = \mathfrak{J}^{\mathcal{Q}_j(\lambda)}(a, g). \tag{10.9}$$

Thus the proof of Theorem 1 shows:

Theorem 7. Let $a \in \mathfrak{A}$, assume $a^2 = I$, and also assume the regularity hypotheses on $Q_j(\lambda)$ and $d_1a = [Q_1(\lambda), a]$, stated in Sect. 11. Then the expectation $\mathfrak{Z}^{Q_j(\lambda)}(a, g)$ is independent of λ .

11. Regularity Hypotheses

As explained in the introduction, our results depend crucially on some regularity hypotheses. In order for **3** to exist, we assume $e^{-H(\lambda)} = e^{-Q(\lambda)^2}$ exists and is trace class on \mathcal{H} . We give sufficient conditions to ensure this, as well as to ensure the validity of the results claimed in Sects. 1–9. The content of Sect. 10 requires only minor modification of these hypotheses. We have explored the consequences of these hypotheses in [QHA].

- 1. The operator Q is self-adjoint on \mathcal{H} , odd with respect to γ , and $e^{-\beta Q^2}$ is trace class for all $\beta > 0$.
- 2. For $\lambda \in \Lambda$, where Λ is an open interval on the real line, the operator $Q(\lambda)$ can be expressed as a perturbation of Q in the form

$$Q(\lambda) = Q + W(\lambda). \tag{11.1}$$

Each $W(\lambda)$ is a symmetric operator on the domain $\mathcal{D} = C^{\infty}(Q)$.

3. Let λ lie in any compact subinterval $\Lambda'\subset\Lambda.$ The inequality

$$W(\lambda)^2 \le aQ^2 + bI, \tag{11.2}$$

holds as an inequality for forms on $\mathcal{D} \times \mathcal{D}$. The constants a < 1 and $b < \infty$ are independent of λ in the compact set $\Lambda' \subset \Lambda$.

4. Let $R = (Q^2 + I)^{-1/2}$. The operator $Z(\lambda) = RW(\lambda)R$ is bounded uniformly for $\lambda \in \Lambda'$, and the difference quotient

$$\frac{Z(\lambda) - Z(\lambda')}{\lambda - \lambda'} \tag{11.3}$$

converges in norm to a limit as λ' → λ ∈ Λ' ⊂ Λ.
5. The bilinear form d_λa satisfies the bound

$$\|R^{\alpha}d_{\lambda}aR^{\beta}\| < M, \tag{11.4}$$

with a constant *M* independent of λ for $\lambda \in \Lambda'$. Here α , β are non-negative constants and $\alpha + \beta < 1$.

In certain examples we are interested in the behavior of $\mathfrak{J}(\lambda, a, g)$ as λ tends to the boundary of Λ . In this case, we may establish the constancy of \mathfrak{J} with estimates that are weaker than (1–5) at the endpoint of Λ , by directly proving the existence and continuity of \mathfrak{J} at the endpoint. We study one such example in [HE], though other types of endpoint singularities are also of interest (often involving a $\lambda \to \infty$ limit).

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